

Math 2010 Week 7

Last time:

Differentiability is defined in terms of linear approximation and error

Ex Show that a linear polynomial

$$f(\vec{x}) = c + b_1 x_1 + \dots + b_n x_n$$

is differentiable on \mathbb{R}^n from definition.

Rmk ① $\frac{\partial f}{\partial x_i}(\vec{x}) = b_i; \forall \vec{x} \in \mathbb{R}^n$

② The linearization of $f(\vec{x})$ at any $a \in \mathbb{R}^n$ is

$$L(\vec{x}) = f(\vec{x})$$

Thm If $f, g: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ are differentiable at $\vec{a} \in \Omega$. Then

- ① $f(\vec{x}) \pm g(\vec{x}), c f(\vec{x}), f(\vec{x}) g(\vec{x})$ are differentiable at \vec{a}
- ② $\frac{f(\vec{x})}{g(\vec{x})}$ is differentiable at \vec{a} if $g(\vec{a}) \neq 0$.

③ (Special case of Chain Rule)

Let $h(x)$ be a one-variable function and is differentiable at $f(\vec{a})$

Then $h \circ f$ is differentiable at \vec{a}

$$\vec{a} \xrightarrow{f} f(\vec{a}) \xrightarrow{h} h \circ f(\vec{a})$$

Rmk We will discuss general case of chain rule later.

Proof of ①, ②, ③ are similar to those for one variable. (MATH 2050)

The results above give many examples of differentiable functions:

• constant functions $f(\vec{x}) = c$

• coordinate functions $f(\vec{x}) = x_i$

• Polynomials (Sum of products of x_i)

eg. $4x^3y^2 + xy^2 - xyz + z^2$ (deg 5)

• Rational functions (Quotient of polynomials)

eg. $\frac{x^3y + z}{x^2 + y^2 + z^2 + 1}$

• If $f(\vec{x})$ is differentiable, then the followings are differentiable:

$$e^{f(\vec{x})}, \sin(f(\vec{x})), \cos(f(\vec{x}))$$

$$\ln(f(\vec{x})) \text{ where } f(\vec{x}) > 0$$

$$|f(\vec{x})| \text{ where } f(\vec{x}) \neq 0$$

$$\sqrt{f(\vec{x})} \text{ where } f(\vec{x}) > 0$$

$$\ln|f(\vec{x})| \text{ where } f(\vec{x}) \neq 0$$

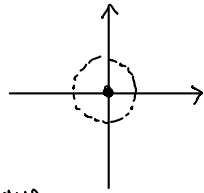
eg.
$$\frac{e^{\sqrt{4 + \sin(x^2 + xy)}}}{\ln(1 + \cos|x^2y|)}$$

Another way to check differentiability (Au 3.5.2)

Thm Let $\Omega \subseteq \mathbb{R}^n$ be open, f be C^1 on Ω
then f is differentiable on Ω

Rmk The assumption requires all $\frac{\partial f}{\partial x_i}$ exist
on an open set, not just at a single point \bar{x}

eg. $f: \mathbb{R}^2 \rightarrow \mathbb{R}$



f_x, f_y exist and are continuous

on a small open disc $D_\epsilon(0,0)$

$\Rightarrow f$ is differentiable on $D_\epsilon(0,0)$

$\Rightarrow f$ is differentiable at $(0,0)$

The theorem provides a simple way to
verify differentiability if all $\frac{\partial f}{\partial x_i}$ can be
easily shown to be continuous

eg $f(x,y,z) = xe^{x+y} - \log(x+z)$

Domain of $f = \{(x,y,z) \in \mathbb{R}^3 : x+z > 0\}$
is open

$$\frac{\partial f}{\partial x} = e^{x+y} + xe^{x+y} - \frac{1}{x+z}$$

$$\frac{\partial f}{\partial y} = xe^{x+y}$$

$$\frac{\partial f}{\partial z} = -\frac{1}{x+z}$$

$\therefore f$ is C^1

$\Rightarrow f$ is differentiable

All are continuous
on domain of f

Pf of thm ($C^1 \Rightarrow$ differentiability on Ω)

We prove it for 2-variable $f(x,y)$. Similar proof for more variables (Ex)

Suppose $(a,b) \in \Omega$ and $B_\delta(a,b) \subseteq \Omega$. For $(x,y) \in B_\delta(a,b)$,

$$\begin{aligned} f(x,y) - f(a,b) &= f(x,y) - f(x,b) + f(x,b) - f(a,b) && \text{(MVT)} \\ &= f_y(x,k)(y-b) + f_x(h,b)(x-a) \text{ for some } k \text{ between } b,y \text{ and } h \text{ between } a,x \end{aligned}$$

$$\begin{aligned} \left| \frac{\mathcal{E}(x,y)}{\|(x,y)-(a,b)\|} \right| &= \left| \frac{f(x,y) - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b)}{\sqrt{(x-a)^2 + (y-b)^2}} \right| \\ &= \left| \frac{[f_y(x,k) - f_y(a,b)](y-b) + [f_x(h,b) - f_x(a,b)](x-a)}{\sqrt{(x-a)^2 + (y-b)^2}} \right| \\ &\leq \left| \frac{[f_y(x,k) - f_y(a,b)](y-b)}{\sqrt{(x-a)^2 + (y-b)^2}} \right| + \left| \frac{[f_x(h,b) - f_x(a,b)](x-a)}{\sqrt{(x-a)^2 + (y-b)^2}} \right| \quad \text{by triangle inequality} \\ &\leq |f_y(x,k) - f_y(a,b)| + |f_x(h,b) - f_x(a,b)| \end{aligned}$$

Take $(x,y) \rightarrow (a,b)$, then $(x,k), (h,b) \rightarrow (a,b) \Rightarrow$ R.H.S $\rightarrow 0$ by continuity of f_x, f_y

By sandwich theorem, $\lim_{(x,y) \rightarrow (a,b)} \left| \frac{\mathcal{E}(x,y)}{\|(x,y)-(a,b)\|} \right| = 0 \Rightarrow f$ is differentiable at (a,b)

Gradient and Directional derivative

Defn Let $\Omega \subseteq \mathbb{R}^n$ be open, $\vec{a} \in \Omega$,
 $f: \Omega \rightarrow \mathbb{R}$. Define the gradient vector
of f at \vec{a} to be

$$\vec{\nabla} f(\vec{a}) = \left(\frac{\partial f}{\partial x_1}(\vec{a}), \dots, \frac{\partial f}{\partial x_n}(\vec{a}) \right)$$

eg $f(x, y) = x^2 + 2xy$

$$\vec{\nabla} f(x, y) = (f_x, f_y) = (2x + 2y, 2x)$$

$$\vec{\nabla} f(1, 2) = (6, 2)$$

Rmk Using $\vec{\nabla} f$, linearization of f at \vec{a}
can be expressed as

$$\begin{aligned} L(\vec{x}) &= f(\vec{a}) + \sum \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) \\ &= f(\vec{a}) + \vec{\nabla} f(\vec{a}) \cdot (\vec{x} - \vec{a}) \end{aligned}$$

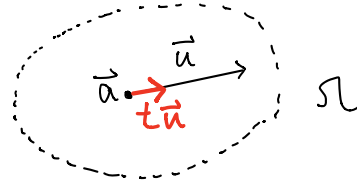
Defn Let $\Omega \subseteq \mathbb{R}^n$ be open, $\vec{a} \in \Omega$, $f: \Omega \rightarrow \mathbb{R}$

Let $\vec{u} \in \mathbb{R}^n$ be a unit vector (i.e. $\|\vec{u}\| = 1$)

Define the directional derivative of f
in the direction of \vec{u} at \vec{a} to be

$$D_{\vec{u}} f(\vec{a}) = \lim_{t \rightarrow 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t}$$

= the rate of change of f in the
direction of \vec{u} at the point \vec{a}



Rmk Let $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$
i-th term

Then $D_{e_i} f(\vec{a}) = \frac{\partial f}{\partial x_i}(\vec{a})$

$$\text{eg } e_2 = (0, 1) \in \mathbb{R}^2$$

$$\begin{aligned} D_{e_2} f(a, b) &= \lim_{t \rightarrow 0} \frac{f(a, b) + t e_2 - f(a, b)}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(a, b+t) - f(a, b)}{t} \\ &= \frac{\partial f}{\partial y}(a, b) \end{aligned}$$

Thm Suppose f is differentiable at \vec{a}

Let $\vec{u} \in \mathbb{R}^n$ be a unit vector. Then

$$D_{\vec{u}} f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{u}$$

Recall that if $\vec{v} \neq \vec{0} \in \mathbb{R}^n$, then

the direction of \vec{v} is defined to be

the unit vector $\frac{\vec{v}}{\|\vec{v}\|}$

$$\text{eg let } f(x, y) = \sin^{-1}\left(\frac{x}{y}\right)$$

Find the rate of change of f at $(1, \sqrt{2})$

in the direction of $\vec{v} = (1, -1)$

$$\text{Sol let } \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$\text{Recall: } (\sin^{-1} z)' = \frac{1}{\sqrt{1-z^2}}$$

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y} \quad \frac{\partial f}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{-x}{y^2}$$

Note f , $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are continuous near $(1, \sqrt{2})$

$\Rightarrow f$ is C^1 near $(1, \sqrt{2})$

$\Rightarrow f$ is differentiable at $(1, \sqrt{2})$

$$\begin{aligned} \therefore \text{Answer} &= D_{\vec{u}} f(1, \sqrt{2}) \\ &= \vec{\nabla} f(1, \sqrt{2}) \cdot \vec{u} \end{aligned}$$

$$= \left(\frac{\partial f}{\partial x}(1, \sqrt{2}), \frac{\partial f}{\partial y}(1, \sqrt{2}) \right) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$$

$$= \left(1, -\frac{1}{\sqrt{2}} \right) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{2}$$

Pf (differentiable $\Rightarrow D_{\vec{u}}f(\vec{a}) = \vec{\nabla}f(\vec{a}) \cdot \vec{u}$)

Let $L(\vec{x})$ be linearization of $f(\vec{x})$ at \vec{a}

$$f(\vec{x}) = L(\vec{x}) + \varepsilon(\vec{x})$$

$$= f(\vec{a}) + \vec{\nabla}f(\vec{a}) \cdot (\vec{x} - \vec{a}) + \varepsilon(\vec{x})$$

Put $\vec{x} = \vec{a} + t\vec{u}$:

$$f(\vec{a} + t\vec{u}) = f(\vec{a}) + \vec{\nabla}f(\vec{a}) \cdot (t\vec{u}) + \varepsilon(\vec{a} + t\vec{u})$$

$$D_{\vec{u}}f(\vec{a}) = \lim_{t \rightarrow 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t}$$

$$= \lim_{t \rightarrow 0} \frac{\vec{\nabla}f(\vec{a}) \cdot (t\vec{u}) + \varepsilon(\vec{a} + t\vec{u})}{t}$$

$$= \vec{\nabla}f(\vec{a}) \cdot \vec{u} + \lim_{t \rightarrow 0} \frac{\varepsilon(\vec{a} + t\vec{u})}{t}$$

Differentiability of f at \vec{a}

$$\Rightarrow \lim_{t \rightarrow 0} \frac{|\varepsilon(\vec{a} + t\vec{u})|}{\|\vec{a} + t\vec{u} - \vec{a}\|} = 0$$

$$\Rightarrow \lim_{t \rightarrow 0} \left| \frac{\varepsilon(\vec{a} + t\vec{u})}{t} \right| = 0$$

By Sandwich theorem,

$$D_{\vec{u}}f(\vec{a}) = \vec{\nabla}f(\vec{a}) \cdot \vec{u} + 0 = \nabla f(\vec{a}) \cdot \vec{u}$$

Geometric Meanings of $\vec{\nabla} f(\vec{a})$

If f is differentiable at \vec{a} , $\|\vec{u}\| = 1$,

$$D_{\vec{u}} f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{u}$$

By Cauchy-Schwarz

$$|\vec{\nabla} f(\vec{a}) \cdot \vec{u}| \leq \|\vec{\nabla} f(\vec{a})\| \|\vec{u}\| = \|\vec{\nabla} f(\vec{a})\|$$

Also, if $\vec{\nabla} f(\vec{a}) \neq \vec{0}$, then

$$-\|\vec{\nabla} f(\vec{a})\| \leq \vec{\nabla} f(\vec{a}) \cdot \vec{u} \leq \|\vec{\nabla} f(\vec{a})\|$$

$$\begin{aligned} \text{"="} \Leftrightarrow \vec{\nabla} f(\vec{a}) = k\vec{u} & \quad \text{"="} \Leftrightarrow \vec{\nabla} f(\vec{a}) = k\vec{u} \\ \text{for some } k < 0 & \quad \text{for some } k > 0 \end{aligned}$$

At \vec{a} , f **increases** (decreases) most rapidly
in the direction of $\vec{\nabla} f(\vec{a})$ ($-\vec{\nabla} f(\vec{a})$)
at a rate of $\|\vec{\nabla} f(\vec{a})\|$

Properties of Gradient

If $f, g: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ are differentiable
open

c is a constant. Then

$$\textcircled{1} \quad \vec{\nabla}(f+g) = \vec{\nabla} f + \vec{\nabla} g$$

$$\vec{\nabla}(cf) = c \vec{\nabla} f$$

$$\textcircled{2} \quad \vec{\nabla}(fg) = g \vec{\nabla} f + f \vec{\nabla} g$$

$$\textcircled{3} \quad \vec{\nabla}\left(\frac{f}{g}\right) = \frac{g \vec{\nabla} f - f \vec{\nabla} g}{g^2} \quad \text{if } g \neq 0$$

Pf Follow easily from partial differentiations.

Rmk In definition of $D_{\vec{u}}f(\vec{a})$,

\vec{u} is assumed to be a unit vector

It can also be generalized to $D_{\vec{v}}f(\vec{a})$

for any \vec{v} (any length).

In that case

$$D_{\vec{v}}f(\vec{a}) = \lim_{t \rightarrow 0} \frac{f(\vec{a} + t\vec{v}) - f(\vec{a})}{t}$$

$$\text{and } D_{\vec{v}}f(\vec{a}) = \vec{\nabla}f(\vec{a}) \cdot \vec{v}$$

Note

$$D_{\vec{v}}f = \begin{cases} \|\vec{v}\| D_{\vec{u}}f & \text{if } \vec{v} \neq \vec{0}, \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} \\ 0 & \text{if } \vec{v} = \vec{0} \end{cases}$$

Total Differential (of a real-valued function)

Let $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable at $\vec{a} \in \Omega$

Consider linearization at \vec{a} :

$$f(\vec{x}) = f(\vec{a}) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) + \mathcal{E}(\vec{x})$$

Denote $\Delta f = f(\vec{x}) - f(\vec{a})$, $\Delta x_i = x_i - a_i$

$$\text{Then } \Delta f \approx \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\vec{a}) \Delta x_i$$

The approximation is good up to 1st order since

$$\lim_{\vec{x} \rightarrow \vec{a}} \frac{\mathcal{E}(\vec{x})}{\|\vec{x} - \vec{a}\|} = 0$$

← 1st order

Classically, this 1st order approximated change is denoted by

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\vec{a}) dx_i$$

and is called the total differential of f at \vec{a}

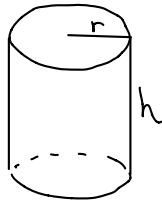
Rmk In more advanced level, df and dx ;
can be interpreted as linear maps from \mathbb{R}^n to \mathbb{R} .

eg Let $V(r, h) = \pi r^2 h$

V is $C^1 \Rightarrow$ differentiable

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

$$= 2\pi r h dr + \pi r^2 dh$$



$V =$ volume of cylinder

For application:

Suppose we want to approximate
change of V when (r, h) changes from
 $(r, h) = (3, 12)$ to $(3 + 0.08, 12 - 0.3)$

$$\text{Let } dr = \Delta r = 0.08,$$

$$dh = \Delta h = -0.3$$

Then $\Delta V \approx dV \leftarrow$ approximated change

$$\begin{aligned} \text{actual change} &= 2\pi r h dr + \pi r^2 dh \\ &= 2\pi(3)(12)(0.08) + \pi(3)^2(-0.3) \\ &= 3.06\pi \approx 9.61 \end{aligned}$$

Properties of total differential

If $f, g: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ are differentiable
open

c is a constant. Then

$$\textcircled{1} \quad d(f+g) = df + dg$$

$$d(cf) = cdf$$

$$\textcircled{2} \quad d(fg) = gdf + fdg$$

$$\textcircled{3} \quad d\left(\frac{f}{g}\right) = \frac{gdf - fdg}{g^2}$$

Pf Follow easily from partial differentiations.

Summary: Differentiating a real-valued function $f(\vec{x}) = f(x_1, \dots, x_n)$ at $\vec{a} \in \mathbb{R}^n$

Different types of derivatives

• Directional derivative: $D_{\vec{u}}f(\vec{a}) = \lim_{t \rightarrow 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t}$ for $\|\vec{u}\| = 1$

• Partial derivative: $\frac{\partial f}{\partial x_i}(\vec{a}) = D_{\vec{e}_i}f(\vec{a})$ $\vec{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$

• Gradient: $\vec{\nabla}f(\vec{a}) = \left(\frac{\partial f}{\partial x_1}(\vec{a}), \dots, \frac{\partial f}{\partial x_n}(\vec{a}) \right)$

• Total differential: $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\vec{a}) dx_i$

• Higher derivatives: eg. $\frac{\partial^2 f}{\partial x_1 \partial x_2} = f_{x_2 x_1}$

f is C^k means f and all its partial derivatives up to order k exist and are continuous

Linear approximation of $f(\vec{x})$ near \vec{a}

$$L(\vec{x}) = f(\vec{a}) + \vec{\nabla}f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

$$f(\vec{x}) = L(\vec{x}) + \varepsilon(\vec{x})$$

• f is differentiable at \vec{a} if $\lim_{\vec{x} \rightarrow \vec{a}} \frac{\varepsilon(\vec{x})}{\|\vec{x} - \vec{a}\|} = 0 \Rightarrow df \approx \Delta f$

Relations among derivatives

$$\textcircled{1} C^\infty \Rightarrow \dots \Rightarrow C^{k+1} \Rightarrow C^k \Rightarrow \dots \Rightarrow C^1 \Rightarrow C^0$$

$\textcircled{2}$ f is C^1 on an open set containing \vec{a}

\Downarrow

f is differentiable at \vec{a}

\Downarrow

$$D_{\vec{u}}f(\vec{a}) = \vec{\nabla}f(\vec{a}) \cdot \vec{u}$$

\Downarrow

$D_{\vec{u}}f(\vec{a})$ exists for any unit vector $\vec{u} \in \mathbb{R}^n$ $\not\Rightarrow$ f is continuous at \vec{a}

\Downarrow

$\frac{\partial f}{\partial x_i}(\vec{a})$ exists for $i=1, \dots, n$

$\not\Rightarrow$

$\textcircled{3}$ All the \Rightarrow in the reverse direction are false. See next page for counter examples

Verify the following (counter-) examples:

eg1 $f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

f is differentiable on \mathbb{R} but

$f'(x)$ is not continuous at $x=0$.

Similarly,

$g(x) = x^{2k-2} f(x)$ is k -time differentiable

but $g^{(k)}(x)$ is not continuous at $x=0$.

Hence, k -time differentiable $\not\Rightarrow C^k$

In particular, $C^{k-1} \not\Rightarrow C^k$

For multivariable, let $h(\vec{x}) = g(x_1)$

eg2

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$D_{\vec{u}} f(0,0)$ exists for any unit vector $\vec{u} \in \mathbb{R}^2$

but f is not continuous at $(0,0)$

eg3 $f(x,y) = |x+y|$

f is continuous on \mathbb{R}^2 but $f_x(0,0), f_y(0,0)$ DNE

eg4 $f(x,y) = \sqrt{|xy|}$

$f_x(0,0), f_y(0,0)$ exist

but $D_{\vec{u}} f(0,0)$ DNE for $\vec{u} \neq \pm \vec{e}_1, \pm \vec{e}_2$.